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PARABOLIC EQUATIONS WITH NONLINEAR SINGULARITIES

PEDRO J. MARTÍNEZ-APARICIO AND FRANCESCO PETITTA

ABSTRACT. We show the existence of a positive solution $u \in L^2(0,T;H_0^1(\Omega))$ for nonlinear parabolic problems with singular lower order terms of asymptotic type. Concretely, we shall consider semilinear problems whose model is

$$\begin{cases} u_t - \Delta u + \dfrac{u}{1-u} = f(x,t) & \text{in } \Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T)\,, \end{cases}$$

and quasilinear problems having natural growth with respect to the gradient, whose model is

$$\begin{cases} u_t - \alpha\Delta u + \dfrac{|\nabla u|^2}{u^\gamma} = f(x,t) & \text{in } \Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T). \end{cases}$$

Moreover, we prove a comparison principle and, as an application, we study the asymptotic behavior of the solution as t goes to infinity.

CONTENTS

1. INTRODUCTION

We study both semilinear problems with an asymptote in the lower order term without any dependence on the gradient and quasilinear boundary value problems with lower order terms having quadratic dependence on the gradient and possesing a singularity at $u = 0$.

Even if it were possible to consider more general singularities we mainly will focus our attention, for the sake of simplicity, on two problems, which turn out to be, in some sense, the extreme cases (see Remark 5.3 below for further details) of a larger variety of problems. Due to the different nature of these problems we shall use completely different techniques to handle with them, trying to give some insights on how the general case could be faced.

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Specifically, we adapt the ideas of the elliptic results in [2],[10] (see also [1],[3],[16]) to prove existence results in the parabolic case. In [10], the author considers an elliptic problem with an asymptote different from zero in the lower order term. On the other hand, the quasilinear elliptic problems for which we extend the results can be seen as a parabolic counterpart of the recent papers [1], [2] and [3]. In these papers the authors prove the existence of a positive solution for problems with lower order term having quadratic gradient and possibly a singularity at zero. Our purpose is to extend some results in [2] to the evolutive case since this type of equations naturally arise in a variety of contexts as stochastic control problems ([8] and [26]), growth patterns in clusters and fronts of solidification (growth of tumors, [18]), flame propagation ([9]) and groundwater flow in a water-absorbing fissurized porous rock ([7]). Moreover, for these types of problems we prove a new comparison principle for parabolic equations following the elliptic framework in [4]. As consequence, we state a uniqueness result for this class of problems and we apply it for establishing a stability result of parabolic solutions toward the stationary solution of the same problem. To make it we readapt some techniques introduced in [28].

Precisely, in the first part of this paper, we study the problem

$$\begin{cases} u_t - \operatorname{div}(M(x,t,u)\nabla u) + g(x,t,u) = f(x,t) & \text{in } \Omega\times(0,T),\\ u(x,0)=u_0(x) & \text{in } \Omega,\\ u(x,t)=0 & \text{on } \partial\Omega\times(0,T), \end{cases} \tag{1.1}$$

where Ω is an open and bounded set of $\mathbb{R}^N$ $(N\geq 3)$, $T>0$, $M(x,t,s)\stackrel{\text{def}}{=}(m_{ij}(x,t,s))$, $i,j=1,\dots,N$ is a symmetric matrix whose coefficients $m_{ij}:\Omega\times(0,T)\times\mathbb{R}\longrightarrow\mathbb{R}$ are Carathéodory functions (i.e., $m_{ij}(\cdot,\cdot,s)$ is measurable on Ω for every $s\in\mathbb{R}$, and $m_{ij}(x,t,\cdot)$ is continuous on $\mathbb{R}$ for a.e. $(x,t)\in\Omega\times(0,T)$) such that there exist constants $0<\alpha\leq\beta$ satisfying

$$\begin{aligned} &\alpha|\xi|^2\leq M(x,t,s)\xi\cdot\xi,\\ &|M(x,t,s)|\leq\beta,\quad \forall(s,\xi)\in\mathbb{R}\times\mathbb{R}^N,\ \text{a.e. } x\in\Omega,\quad \forall t\in(0,T)\,. \end{aligned}\tag{1.2}$$

We consider a nonnegative function $f\in L^1(\Omega\times(0,T))$, $\kappa>0$ and $g:\Omega\times(0,T)\times[0,\kappa)\longrightarrow\mathbb{R}^+$ a Carathéodory function such that

$$h(s)\leq g(x,t,s)\leq\rho(x,t)\delta(s),\quad \forall s\in[0,\kappa),\ \text{a.e. } x\in\Omega,\quad \forall t\in(0,T) \tag{1.3}$$

where $0\leq\rho\in L^1(\Omega\times(0,T))$ and $\delta(s),h(s):[0,\kappa)\longrightarrow\mathbb{R}^+$ are continuous and increasing real functions such that $h(0)=0$ and $\lim_{s\to\kappa^-}h(s)=+\infty$.

Observe, that the nonlinear term g has an asymptote in κ. Due to the structure of the nonlinearity g it is natural to consider initial data u_0 which are measurable and strictly less than κ almost everywhere on Ω. In what follows we denote $Q:=\Omega\times(0,T)$.

Let us specify that a solution of problem (1.1) is a nonnegative function $u\in L^2(0,T;H_0^1(\Omega))\cap C([0,T];L^1(\Omega))$, such that $u<\kappa$ a.e. on Q, $g(x,t,u)$ belongs to

$L^1(Q)$ and $u(x,0)=u_0<\kappa$ which satisfies

$$-\int_\Omega u_0\varphi(0)-\int_0^T\langle\varphi_t,u\rangle+\int_Q M(x,t,u)\nabla u\nabla\varphi$$
$$+\int_Q g(x,t,u)\varphi=\int_Q f(x,t)\varphi,$$

for any $\varphi\in L^2(0,T;H_0^1(\Omega))\cap L^\infty(Q)$ with $\varphi_t\in L^2(0,T;H^{-1}(\Omega))$ and $\varphi(T)=0$.

Concretely, our first result is the following.

Theorem 1.1. *Let $f\in L^1(Q)$ be nonnegative, assume that M satisfies* (1.2) *and g verifies* (1.3)*, then problem (1.1) admits a solution in $L^2(0,T;H_0^1(\Omega))$.*

The proof of Theorem 1.1 will be based on an *double* approximation argument. If $\|u_0\|_{L^\infty(\Omega)}<\kappa$ then we readapt the argument of [10] in order to pass to the limit in the approximation problem. Then, to get rid of the general case of initial data possibly *touching* the singular value κ, we perform a truncation argument. Notice that, as far as the problem is concerned, the main task in order to prove the result is the proof of strongly compactness in $L^1(Q)$ of the approximating lower order terms.

The second part of this paper will be mainly devoted to the study of problems having a singular lower order term with natural growth with respect to the gradient

$$\begin{cases} u_t-\operatorname{div}(M(x,t,u)\nabla u)+g(x,t,u)|\nabla u|^2=f(x,t) & \text{in } Q,\\ u(x,0)=u_0(x) & \text{in } \Omega,\\ u(x,t)=0 & \text{on } \partial\Omega\times(0,T), \end{cases} \tag{1.4}$$

where Ω is an open and bounded set in $\mathbb{R}^N$ $(N\ge 3)$, M satisfies (1.2) and $f\in L^r(0,T;L^q(\Omega))$ with $\dfrac{1}{r}+\dfrac{2}{Nq}<1$, $q\ge1$, $r>1$ satisfies

$$m_\omega(f)=\operatorname{ess\,inf}\{f(x,t):x\in\omega,\ t\in(0,T)\}>0,\quad \forall\omega\subset\subset\Omega\,. \tag{1.5}$$

Moreover, we consider initial data $u_0\in L^\infty(\Omega)$ and we suppose that

$$m_\omega(u_0)=\operatorname{ess\,inf}\{u_0(x):x\in\omega\}>0,\quad \forall\omega\subset\subset\Omega\,. \tag{1.6}$$

Concerning the lower order term, we assume that the function $g(x,t,s)$ satisfies for some $\mu>0$ that

$$-\frac{\mu}{s}\le g(x,t,s)\le h(s),\quad \text{for a.e. } x\in\Omega,\ \forall s>0,\quad \forall t\in(0,T) \tag{1.7}$$

where $h:(0,+\infty)\to[0,+\infty)$ is a continuous nonnegative function such that

$$\lim_{s\to0^+}\int_s^1\sqrt{h(t)}\,dt<+\infty, \tag{1.8}$$
and $h(s)$ is nonincreasing in a neighborhood of zero.

Observe that, if $g(x,t,\cdot)$ is bounded and continuous on $\mathbb{R}$, then problem (1.4) was largely studied in the past with many results concerning existence, nonexistence and regularity of the solution depending on the regularity of the data and on the growth of g at infinity (see [11], [15], [29], and references therein).

Let us stress that, because of the possibly singularity in 0, then problem (1.4) turn out to be, in some sense, *much more singular* than the previous one. We look for a solution which is zero at the boundary of the cylinder $\partial\Omega \times (0,T)$ while the singularity in zero would contrast this fact. This is because, to handle with this problem, we need a completely different approach. Namely, we argue by localizing the problem, and then we look at *how much the lower order term can be singular* to ensure the existence of a solution.

Finally notice that, as far as condition (1.7) is concerned, it allows to the lower order term to have a singularity at $s = 0$ and to change of sign. The nonlinearity considered in the model problem is $g(x,t,s) = \dfrac{1}{s^\gamma}$. In this case, hypothesis (1.8) holds provided that $\gamma < 2$.

Let us specify that, for the sake of exposition, we have chosen to handle with data $f \in L^r(0,T;L^q(\Omega))$ with $\dfrac{1}{r} + \dfrac{2}{Nq} < 1$, $q \geq 1$, $r > 1$ and $u_0 \in L^\infty(\Omega)$. In fact, the main idea of our proofs are flexible enough to be extended to more general data, namely $f \in L^1(Q)$, satisfying (1.5), and u_0 a nonnegative function satisfying

$$\varsigma(x) = \int_{u_0}^1 e^{-\frac{H(t)}{\alpha}}\,dt \in L^1_{\mathrm{loc}}(\Omega),$$

where H is the primitive of the function h that appears in (1.3) (the meaning of this condition can be easily deduced by the proof of Theorem 1.3 in Section 3). This can be handled by using standard approximation/compactness arguments (see for instance [2] and [12]). However, these arguments would force us to handle with infinite energy solutions. We prefer to avoid technicalities to focus on the core of the problem which is the presence of the *singular term* $g(x,t,u)|\nabla u|^2$ in a Cauchy problem with *homogeneous* boundary conditions.

A solution of problem (1.4) is a function $u \in L^2(0,T;H_0^1(\Omega)) \cap C([0,T];L^1(\Omega))$, such that $u > 0$ a.e. on Q, $g(x,t,u)|\nabla u|^2$ belongs to $L^1(Q)$ and $u(x,0) = u_0$ a.e. on Ω which satisfies

$$-\int_\Omega u_0\varphi(0) - \int_0^T \langle \varphi_t, u\rangle + \int_Q M(x,t,u)\nabla u\nabla\varphi$$
$$+\int_Q g(x,t,u)|\nabla u|^2\varphi = \int_Q f\varphi,$$

for any $\varphi \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(Q)$ with $\varphi_t \in L^2(0,T;H^{-1}(\Omega))$ and $\varphi(T) = 0$.

Remark 1.2. Let us observe that in both definitions of solution we have imposed the technical condition $\varphi(T) = 0$. An easy density argument allow us to show that, in fact, we are allowed to take also test functions which do not vanish in T provided by a suitably modify the definition. Concretely, we have to substitute the integrals involving the time derivative of the test function

$$-\int_\Omega u_0\varphi(0) - \int_0^T \langle \varphi_t, u\rangle,$$

with

$$\int_\Omega u(T)\varphi(T) - \int_\Omega u_0\varphi(0) - \int_0^T \langle \varphi_t, u\rangle\,.$$

Notice that, because of the fact that $u \in C([0,T],L^1(\Omega))$, all terms in the above expression are well defined. We will made use of this fact to prove our comparison and uniqueness results.

Our result concerning existence and regularity of a solution for problem (1.4) is the following.

Theorem 1.3. *Let* $f \in L^r(0,T;L^q(\Omega))$ *with* $\dfrac{1}{r}+\dfrac{2}{Nq}<1$, $q\geq 1$, $r>1$ *satisfying* (1.5), $u_0 \in L^\infty(\Omega)$ *satisfying (1.6), and assume that* (1.2), (1.7) *and* (1.8) *hold. If* $\alpha>\mu$ *then problem* (1.4) *admits a solution in* $L^2(0,T;H_0^1(\Omega))\cap L^\infty(Q)$.

The proof of this result will be based on an approximation and compactness argument where the key role is played by an uniform local estimate from below of the approximating sequence of solutions. Furthermore, we prove a comparison principle that will be essential to study the large time behavior of the solutions. The basic idea to prove the comparison principle is to take a test function (see [4]) that state an inequality where the quadratic term on the gradient is cancelled out. Using some techniques of [28] we handle with the asymptotic behavior of the solutions as t goes to infinity.

The plan of the paper is the following: in Section 2 we prove Theorem 1.1. In Section 3 we prove Theorem 1.3, and we give an account on some comparison and large time behavior results concerning for problem (1.1). Section 4 is devoted to a comparison result for (1.4). Moreover, we establish a uniqueness result for problem (1.4) and in Section 5 we prove the stability of solutions of problem (1.4) toward the stationary solution of the problem.

2. Parabolic semilinear problems with asymptotes

In this section we prove Theorem 1.1 and we give some remarks on uniqueness and asymptotic behavior of the solutions.

Proof. We divide the proof in two steps.

Step 1. We prove Theorem 1.1 for a bounded datum $f \in L^\infty(Q)$ and $\|u_0\|_{L^\infty(\Omega)}<\kappa$.

Step 2. By an approximating argument, we use Step 1 for proving Theorem 1.1.

Step 1. Denote by T_k and G_k the truncatures function defined, respectively, as

$$T_k(s)=\begin{cases} -k & \text{if } s\leq -k,\\ s & \text{if } -k<s<k,\\ k & \text{if } k\leq s\end{cases}$$

and $G_k(s)=s-T_k(s)$ for every $s\in\mathbb{R}$.

Let us define $g_n(x,t,s)=T_n(g(x,t,s))$. We consider the following approximated problem

$$\begin{cases} (u_n)_t-\operatorname{div}(M(x,t,u_n)\nabla u_n)+g_n(x,t,u_n)=f(x,t) & \text{in } Q,\\ u_n(x,0)=u_0(x) & \text{in } \Omega,\\ u_n(x,t)=0 & \text{on } \partial\Omega\times(0,T).\end{cases}$$

By [25] there exists a solution $u_n\in L^2(0,T;H_0^1(\Omega))\cap C([0,T];L^2(\Omega))$ of the above problem and $(u_n)_t\in L^2(0,T;H^{-1}(\Omega))$. In addition, thanks to [6] there exists $l>0$ (independent on n) such that $\|u_n\|_{L^\infty(Q)}\leq l$, for every $n\in\mathbb{N}$.

Now, we are going to prove that u_n are a priori bounded in both spaces $L^2(0,T;H_0^1(\Omega))$ and $L^\infty(0,T;L^2(\Omega))$. Indeed, if we use $\varphi = u_n$ as test function in the approximated problem, it follows that

$$\int_0^T \langle (u_n)_t, u_n \rangle + \int_Q M(x,t,u_n)|\nabla u_n|^2 + \int_Q g_n(x,t,u_n)u_n = \int_Q f u_n.$$

Since

$$\int_0^T \langle (u_n)_t, u_n \rangle = \frac{1}{2}\int_Q \frac{d}{dt}u_n^2 = \frac{1}{2}\int_\Omega u_n^2(T) - \frac{1}{2}\int_\Omega u_n^2(0) \tag{2.1}$$

and using (1.2), the boundedness of u_n and Young's inequality we get

$$\frac{1}{2}\int_\Omega u_n^2(T) + \alpha\int_Q |\nabla u_n|^2 + \int_Q g_n(x,t,u_n)u_n$$

$$\leq l\int_Q f + \frac{1}{2}\|u_0\|^2_{L^2(\Omega)}.$$

This implies that $\{u_n\}$ is bounded in $L^2(0,T;H_0^1(\Omega))$ (in particular, up to a subsequence, $u_n \rightharpoonup u$ in $L^2(0,T;H_0^1(\Omega))$) and in $L^\infty(0,T;L^2(\Omega))$. Moreover, $g_n(x,t,u_n)u_n$ is bounded in $L^1(Q)$.

Notice that, since $(u_n)_t$ is uniformly bounded in $L^2(0,T;H^{-1}(\Omega)) + L^1(Q)$, then we can use the classic Aubin-Simon compactness arguments (see Corollary 4 in [30]) to deduce the almost everywhere convergence of u_n toward u.

Let us consider $\eta \equiv \max\left\{h^{-1}(\|f\|_{L^\infty(Q)}), \|u_0\|_{L^\infty(\Omega)}\right\} < \kappa$ and $\theta(s) = \displaystyle\int_0^s (r-\eta)^+\,dr$. Since $u_0 < \kappa$, we get

$$\int_0^T \left\langle (u_n)_t, (u_n-\eta)^+ \right\rangle = \int_Q \frac{d}{dt}\theta(u_n)$$

$$= \int_\Omega \theta(u_n(T)) - \int_\Omega \theta(u(0)) \geq -\int_\Omega \theta(u_0) = 0,$$

and using $\varphi = (u_n-\eta)^+$ as test function in the approximated problem, (1.2) and (1.3) we deduce that

$$\int_Q \left[T_n(h(u_n)) - f(x,t)\right](u_n-\eta)^+ \leq \int_Q \left[g_n(x,t,u_n) - f(x,t)\right](u_n-\eta)^+ \leq 0,$$

i.e.

$$0 \geq \int_{\{n>h(u_n)\geq h(\eta)\}} \left[h(u_n) - f(x,t)\right](u_n-\eta)^+$$

$$+\int_{\{h(u_n)\geq n\geq h(\eta)\}} \left[n - f(x,t)\right](u_n-\eta)^+ .$$

Observing that clearly the right-hand of this inequality is nonnegative we have that

$$0 \leq u_n \leq \eta.$$

Notice that in particular, using the above inequality we get

$$0 \le u \le \eta.$$

We proceed now to pass to the limit in the approximated problem. We follow the ideas in [10]. Using the integration by parts and the weak convergence of u_n to u in $L^2(0,T;H_0^1(\Omega))$ we readily have, for any $\varphi \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(Q)$ with $\varphi_t \in L^2(0,T;H^{-1}(\Omega))$ and $\varphi(T)=0$,

$$\int_0^T \langle (u_n)_t, \varphi\rangle = -\int_\Omega u_0\varphi(0) - \int_0^T \langle \varphi_t, u_n\rangle \to -\int_\Omega u_0\varphi(0) - \int_0^T \langle \varphi_t, u\rangle.$$

On the other hand, the weak convergence of u_n to u in $L^2(0,T;H_0^1(\Omega))$, the a.e. and the $*$-weak convergence of $M(x,t,u_n)$ to $M(x,t,u)$ in $L^\infty(Q)$ implies that

$$\int_Q M(x,t,u_n)\nabla u_n \cdot \nabla\varphi \to \int_Q M(x,t,u)\nabla u \cdot \nabla\varphi.$$

Now we prove the equiintegrability of the sequence $\{g_n(x,t,u_n)\}$.
For any measurable subset E of Q, we have

$$\int_E g_n(x,t,u_n(x,t)) = \int_{E\cap\{u_n(x,t)<\eta\}} g_n(x,t,u_n(x,t))$$
$$\le \int_{E\cap\{u_n(x,t)<\eta\}} g(x,t,u_n(x,t)) \le \delta(\eta)\int_E \rho(x,t).$$

And so

$$\lim_{\mathrm{meas}(E)\to 0} \int_E g_n(x,t,u_n(x)) = 0.$$

The above equiintegrability of $g_n(x,t,u_n(x,t))$ and the a.e. convergence to $g(x,t,u(x,t))$ imply that

$$g_n(x,t,u_n) \to g(x,t,u) \quad \text{in } \ L^1(Q).$$

Thus, we can pass to the limit in the sequence of approximating problems to obtain that u is a solution of (1.1) with $f \in L^\infty(Q)$ and $\|u_0\|_{L^\infty(\Omega)} < \kappa$.

Step 2. We consider $f_n = T_n(f)$, and the solutions of the approximated problem

$$\begin{cases} (u_n)_t - \operatorname{div}(M(x,t,u_n)\nabla u_n) + g(x,t,u_n) = f_n(x,t) & \text{in } Q,\\ u_n(x,0) = T_{\kappa-\frac{1}{n}}(u_0(x)) & \text{in } \Omega,\\ u_n(x,t) = 0 & \text{on } \partial\Omega\times(0,T), \end{cases}$$

that turn out to exist thanks to the previous step. We also have that

$$0 \le u_n \le \eta_n = \max\left\{h^{-1}(\|f_n\|_{L^\infty(Q)}), \|u_0\|_{L^\infty(\Omega)}\right\} < \kappa\,.$$

Arguing as in the previous step we have

$$\int_0^T \langle (u_n)_t, u_n\rangle + \alpha\int_Q |\nabla u_n|^2 + \int_Q g(x,t,u_n)u_n \le \int_Q f_n u_n$$

and hence we obtain the estimates in $L^2(0,T;H_0^1(\Omega))$, in $L^\infty(0,T;L^2(\Omega))$ and that $g_n(x,t,u_n)u_n$ is bounded in $L^1(Q)$.

If, for $s>0$ and $\varepsilon>0$ such that $s+\varepsilon<\kappa$, we take $T_{s+\varepsilon}(G_s(u_n))$ as test function in the approximated problem. Using that $0 \le u_n \le \eta_n$ and dropping positive terms, we deduce that

$$\int_Q g(x,t,u_n(x,t))T_{s+\varepsilon}(G_s(u_n(x,t))) \le \int_{\{s\le u_n(x,t)<\kappa\}} f_n + \int_{\{s\le u_0<\kappa\}} u_0, \quad \forall s<\kappa.$$

By virtue of the sign condition on g we can apply Fatou lemma to obtain, by taking limits as ε tends to zero,

$$\int_{\{s\le u_n(x,t)\}} g(x,t,u_n(x,t)) \le \int_{\{s\le u_n(x,t)\}} f_n + \int_{\{s\le u_0\}} u_0, \quad \forall s<\kappa.$$

Since $f_n \le f$, it follows that

$$\int_{\{s\le u_n(x,t)\}} g(x,t,u_n(x,t)) \le \int_{\{s\le u_n(x,t)\}} f + \int_{\{s\le u_0\}} u_0, \quad \forall s<\kappa. \tag{2.2}$$

Therefore, for any measurable subset E of Q, we have

$$\begin{aligned}
&\int_E g_n(x,t,u_n(x,t)) \\
&= \int_{E\cap\{0\le u_n(x,t)<s\}} g(x,t,u_n(x,t)) + \int_{E\cap\{s\le u_n(x,t)\}} g(x,t,u_n(x,t)) \\
&\le \int_{\{s\le u_n(x,t)\}} f + \int_{\{s\le u_0\}} u_0 + \delta(s)\int_E \rho(x,t), \quad \forall s<\kappa.
\end{aligned} \tag{2.3}$$

On the other hand, from (1.3), since $g(x,t,u_n(x,t))u_n(x,t)$ is bounded in $L^1(Q)$, we have

$$h(s)s\int_{\{s\le u_n(x,t)\}} dx \le \int_{\{s\le u_n(x,t)\}} h(u_n(x,t))u_n(x,t) \le \int_{\{s\le u_n(x,t)\}} g(x,t,u_n(x,t))u_n(x,t)$$

$$\le \int_Q g(x,t,u_n(x,t))u_n(x,t) \le L.$$

So that, using that $\lim\limits_{s\to\kappa^-} h(s)s = +\infty$, we obtain

$$\lim_{s\to\kappa} \operatorname{meas}\{(x,t) : s\le u_n(x,t)\} = 0.$$

Since $f\in L^1(Q)$, and $u_0<\kappa$, it follows that, for any fixed $\varepsilon>0$, there exists $0<s_0<\kappa$ such that

$$\int_{\{s_0\le u_n(x,t)<\kappa\}} f<\varepsilon, \quad \int_{\{s_0<u_0<\kappa\}} u_0<\varepsilon.$$

By the absolutely continuity of the integral, we conclude from (2.3) that

$$\lim_{\operatorname{meas}(E)\to 0}\int_E g(x,t,u_n(x,t)) \le \varepsilon.$$

Thus, we have proved that $\{g_n(x,t,u_n)\}$ is equiintegrable. Hence, by Vitali's Theorem we obtain that

$$g(x,t,u_n) \to g(x,t,u) \text{ strongly in } L^1(Q).$$

Therefore, if we pass to the limit as $n \to \infty$ in the notion of weak solution of the approximated problem

$$-\int_\Omega u_n(0)\varphi(0) - \int_0^T \langle \varphi_t, u_n \rangle + \int_Q M(x,t,u_n)\nabla u_n \cdot \nabla\varphi$$

$$+ \int_Q g(x,t,u_n)\varphi = \int_Q f_n \varphi,$$

we obtain that

$$-\int_\Omega u_0\varphi(0) - \int_0^T \langle \varphi_t, u \rangle + \int_Q M(x,t,u)\nabla u \nabla\varphi$$

$$+ \int_Q g(x,t,u)\varphi = \int_Q f\varphi,$$

for any $\varphi \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(Q)$ with $\varphi_t \in L^2(0,T;H^{-1}(\Omega))$. Therefore, Theorem 1.1 is proved. Let us observe that, by applying Fatou lemma in (2.2), we deduce that

$$\int\limits_{\{s\le u(x,t)<\kappa\}} g(x,t,u(x,t)) \le \int\limits_{\{s<u(x,t)<\kappa\}} f + \int\limits_{\{s<u_0<\kappa\}} u_0.$$

□

Remark 2.1. If $g(x,t,\cdot)$ is nondecreasing function then we can easily check uniqueness in a quite standard way by taking the difference of the Landes regularization (see (3.6) below for its definition) of two solutions w_1, w_2, which is an admissible test function, in the weak formulation of problem (1.1), to obtain that $w_1 \equiv w_2$. The comparison between subsolution $\underline{w}$ and supersolutions $\overline{w}$, follows in the same way by testing the equation with suitable regularization of $(\underline{w} - \overline{w})^+$.

We can now state the result about the asymptotic behavior of the solutions as t goes to infinity. We shall give a sketch of the proof in Section 5.

Theorem 2.2. *Let be $0 \le f \in L^1(\Omega)$, $f \not\equiv 0$ and $u_0 < \kappa$ be a nonnegative function. Let M and g satisfying, respectively, (1.2) and (1.3). Moreover, let both M and g be independent of t. If g is nondecreasing, then $u(x,t)$, the weak solution of problem*

$$\begin{cases} u_t - \operatorname{div}(M(x,u)\nabla u) + g(x,u) = f(x) & \text{in } \Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \end{cases}$$

satisfies

$$\lim_{t\to+\infty} u(x,t) = v(x) \qquad \text{a.e. and } *\text{-weakly } L^\infty(\Omega),$$

where v is the unique solution of the stationary problem

$$\begin{cases} -\operatorname{div}(M(x,v)\nabla v) + g(x,v) = f(x) & \text{in } \Omega, \\ v(x) = 0 & \text{on } \partial\Omega\,. \end{cases}$$

3. Parabolic quasilinear problems with singular lower order terms and natural growth with respect to the gradient

In this section we prove Theorem 1.3. The idea to prove it consists in approximating (1.4) by a sequence of problems which fall into the framework of [25] and to prove that their solutions u_n converge to a positive solution of (1.4). We define the Carathéodory function g_n in Q by

$$g_n(x,t,s) = \begin{cases} 0 & \text{if } s \le 0, \\ n^2 s^2 T_n(g(x,t,s)) & \text{if } 0 < s < \frac{1}{n}, \\ T_n(g(x,t,s)) & \text{if } \frac{1}{n} \le s. \end{cases}$$

Observe that g_n verifies

$$g_n(x,t,s) \le T_n(g(x,t,s)) \le g(x,t,s), \tag{3.1}$$

a.e. $x \in \Omega$, $t \in (0,T)$, $s \in \mathbb{R}^+$. By (1.7), we also have

$$g_n(x,t,s)s + \mu \ge 0 \quad \text{a.e. } x \in \Omega, \quad \forall t \in (0,T), \quad \forall s \in \mathbb{R}^+, \tag{3.2}$$

for every $n \in \mathbb{N}$. Therefore,

$$s g_n(x,t,s) \frac{|\xi|^2}{1 + \frac{1}{n}|\xi|^2} + \mu|\xi|^2 \ge 0, \tag{3.3}$$

for a.e. $x \in \Omega$, for every $t \in (0,T)$, $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$.

By [25], Theorem 3.1 (see also [24], Theorem 2.1) there exists a solution $u_n \in L^2(0,T;H_0^1(\Omega))$ of the approximated problem

$$\begin{cases} (u_n)_t - \operatorname{div}(M(x,t,u_n)\nabla u_n) + g_n(x,t,u_n)\dfrac{|\nabla u_n|^2}{1+\frac{1}{n}|\nabla u_n|^2} = f(x,t) & \text{in } Q, \\ u_n(x,0) = u_0(x) & \text{in } \Omega, \\ u_n(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{3.4}$$

with $(u_n)_t \in L^2(0,T;H^{-1}(\Omega))$. Moreover, there exists $d > 0$ (independent on n) such that $\|u_n\|_{L^\infty(Q)} \le d$ for every $n \in \mathbb{N}$ (see for instance [6]).

We prove that $\{u_n\}$ are a priori bounded in both spaces $L^2(0,T;H_0^1(\Omega))$ and $L^\infty(0,T;L^2(\Omega))$. Taking $\varphi = u_n$ as test function in (3.4) and using (1.2) it follows that

$$\int_0^T \langle (u_n)_t, u_n \rangle + (\alpha - \mu)\int_Q |\nabla u_n|^2 + \int_Q g_n(x,t,u_n)\frac{|\nabla u_n|^2}{1+\frac{1}{n}|\nabla u_n|^2} u_n$$

$$+\mu|\nabla u_n|^2 \le \int_Q f u_n \le \|f\|_{L^2(0,T;H^{-1}(\Omega))}\|u_n\|_{L^2(0,T;H_0^1(\Omega))}.$$

Then, by (2.1) and Young's inequality we obtain

$$\frac{1}{2}\int_\Omega u_n^2(T) + \frac{\alpha-\mu}{2}\int_Q |\nabla u_n|^2 + \int_Q g_n(x,t,u_n)\frac{|\nabla u_n|^2}{1+\frac{1}{n}|\nabla u_n|^2} u_n + \mu|\nabla u_n|^2$$

$$\le C_{\alpha,\mu}\|f\|^2_{L^2(0,T;H^{-1}(\Omega))} + \frac{1}{2}\|u_0\|^2_{L^2(\Omega)}.$$

Notice that, since $\alpha > \mu$ we get that

- $\{u_n\}$ is bounded in $L^2(0,T;H_0^1(\Omega))$ (in particular, up to a subsequence, $u_n \rightharpoonup u$ in $L^2(0,T;H_0^1(\Omega))$).
- $g_n(x,t,u_n)\dfrac{|\nabla u_n|^2}{1+\frac{1}{n}|\nabla u_n|^2}u_n$ is bounded in $L^1(Q)$.

Taking also $u_n^- \equiv \min\{u_n,0\}$ as test function in (3.4) and using again (1.2), (2.1) and (3.3) we obtain

$$\frac{1}{2}\int_\Omega u_n^2(t) - \frac{1}{2}\int_\Omega (u_0^-)^2 + (\alpha-\mu)\int_Q |\nabla u_n^-|^2 \leq 0,$$

and, thanks to the positivity of the two first terms (recall that $u_0 > 0$) we have

$$(\alpha-\mu)\int_Q |\nabla u_n^-|^2 \leq 0.$$

Since $\alpha > \mu$, we deduce that $u_n \geq 0$ a.e. in Ω.

Even more, we prove that $u_n > 0$ in Q. Indeed, let $C_n > 0$ be such that $g_n(x,t,s) \leq C_n s$, for $s \in [0,d]$. Therefore u_n satisfies

$$\begin{aligned}
&(u_n)_t - \operatorname{div}(M(x,t,u_n)\nabla u_n) + nC_n u_n \\
&\geq (u_n)_t - \operatorname{div}(M(x,t,u_n)\nabla u_n) + g_n(x,t,u_n)\frac{|\nabla u_n|^2}{1+\frac{1}{n}|\nabla u_n|^2} \\
&= f\,,
\end{aligned}$$

in the sense of distributions and since f is nonnegative and not identically zero, by the strong maximum principle (see [5] for instance) we deduce that $u_n > 0$ in Q.

Proof of Theorem 1.3. The proof will be concluded by proving the following steps:
Step 1. The solutions of the approximated problem are uniformly away from zero in every subset $\omega \times (0,T)$ of Q with $\omega \subset\subset \Omega$.
Step 2. Strong convergence of the approximating solutions in $L^2(0,T;H^1(\omega))$.
Step 3. Passing to the limit in (3.4).

Step 1. For $s > 0$, we define the nondecreasing function

$$H(s) = \int_1^s \widetilde{h}(t)dt = \int_1^s h(t)dt + \log s^\alpha\,,$$

where $\widetilde{h}(s) = h(s) + \dfrac{\alpha}{s}$, and we then consider the nonincreasing function

$$\psi(s) = \int_s^1 \mathrm{e}^{-\frac{H(t)}{\alpha}}dt\,,$$

defined through $H(s)$. Now, we perform the change of variable $v_n := \psi(u_n)$.

Observe that it is well-defined since $\lim_{s\to 0^+}\psi(s) = +\infty$ and $\lim_{s\to+\infty}\psi(s) = \psi_\infty \in [-\infty, 0)$.

We claim that u_n is bounded away from zero (with the bound possibly depending on n) in every open subset $\omega\times(0,T)$ of Q with ω compactly embedded in Ω. Indeed, u_n is continuous (see for instance [14] and [20]) and, as we proved before, strictly positive in Q.

Now, by the chain rule, we have

$$\nabla v_n = -\mathrm{e}^{-\frac{H(u_n)}{\alpha}}\nabla u_n \in L^2(0,T;L^2(\omega)), \quad \forall \omega \subset\subset \Omega,$$

and thus $v_n \in L^2(0,T;H^1(\omega))$ for every $\omega \subset\subset \Omega$.

We are now in position to take $\mathrm{e}^{-\frac{H(u_n)}{\alpha}}\phi$ with $0 \le \phi \in C_c^\infty(\Omega)$ as test function in (3.4) to deduce from the inequality $h(s) \le \widetilde{h}(s)$ and from (1.7), that

$$\int_0^T \langle (u_n)_t, \mathrm{e}^{-\frac{H(u_n)}{\alpha}}\phi\rangle - \int_Q M(x,t,u_n)\nabla u_n \cdot \nabla u_n \frac{\widetilde{h}(u_n)}{\alpha}\mathrm{e}^{-\frac{H(u_n)}{\alpha}}\phi$$

$$+ \int_Q M(x,t,u_n)\nabla u_n \cdot \nabla\phi \, \mathrm{e}^{-\frac{H(u_n)}{\alpha}} + \int_Q \widetilde{h}(u_n)|\nabla u_n|^2 \mathrm{e}^{-\frac{H(u_n)}{\alpha}}\phi$$

$$\ge \int_Q f\mathrm{e}^{-\frac{H(u_n)}{\alpha}}\phi\,.$$

Applying (1.2) together with we definition of ψ, we get

$$-\int_0^T \langle \psi(u_n)_t, \phi\rangle - \int_Q M(x,t,u_n)\nabla\psi(u_n)\cdot\nabla\phi \ge \int_Q f\mathrm{e}^{-\frac{H(u_n)}{\alpha}}\phi$$
$$\ge \int_Q \left(\mathrm{e}^{-\frac{H(u_n)}{\alpha}} - 1\right) f\phi\,.$$

We call $\widetilde{M}(x,t,s) = M(x,t,\psi^{-1}(s))$ and $b(s) = \mathrm{e}^{-\frac{H(s)}{\alpha}} - 1$ for every $s \in (\psi_\infty, +\infty)$. Thus, we deduce that v_n is subsolution of

$$z_t - \operatorname{div}(\widetilde{M}(x,t,z)\nabla z) + f(x,t)\,b(z) = 0, \quad \text{in } Q.$$

As it is proved in [2], $b(s)$ verifies the well-known Keller-Osserman condition (see [19], [27] and [31] for instance) thanks to (1.7) and (1.8). Since both f satisfies (1.5) and u_0 satisfies (1.6), we can apply Lemma 3.12 in [22] to the previous equation to obtain that there exists $C_{\omega,T} > 0$ such that

$$v_n \le C_{\omega,T}, \quad \forall x \in \omega \text{ and } \forall t \in (0,T).$$

Therefore, we obtain that there exists $c_{\omega,T} > 0$ (independent on n) such that

$$u_n \ge \psi(C_0) = c_{\omega,T}, \quad \text{in } \omega \times (0,T).$$

Step 2. Local strong convergence of the approximated solutions.

From (3.4) we obtain that $\{(u_n)_t\}$ is bounded in $L^2(0,T;H^{-1}(\Omega)) + L^1(\omega \times (0,T))$. Using Aubin-Simon compactness arguments (see again Corollary 4 in [30]) we have

$$u_n \to u \text{ in } L^2(\omega \times (0,T)).$$

Now, we prove that for every $\omega \subset\subset \Omega$,

$$u_n \to u \quad \text{in } L^2(0,T;H^1(\omega)). \tag{3.5}$$

We introduce a time-regularization of functions u belonging to $L^2(0,T;H_0^1(\Omega))$ (see [21]): given $\nu > 0$, we define

$$u_\nu(x,t) = \nu\int_{-\infty}^t \tilde{u}(x,s)e^{\nu(s-t)}ds + e^{-\nu t}u_0, \tag{3.6}$$

where $\tilde{u}(x,s)$ is the zero extension of u for $s \notin [0,T]$. From now on, the letter ν will be only used with this meaning. We recall that u_ν converges to u strongly in $L^2(0,T;H_0^1(\Omega))$ as ν tends to infinity, and that $\|u_\nu\|_{L^q(\Omega)} \leq \|u\|_{L^q(\Omega)}$ for every $q \in [1,+\infty]$; moreover,

$$u_\nu(x,0) = u_0 \quad \text{and} \quad (u_\nu)_t = \nu(u - u_\nu),$$

in the sense of distributions (see [21] for the proof of these properties). Observe that, if $u \in L^\infty(Q)$, then by the last property the derivative of u_ν with respect to time belongs to $L^\infty(Q) \subset L^2(0,T;H^{-1}(\Omega))$, and therefore

$$\langle (u_\nu)_t, \phi\rangle = \nu \int_Q (u - u_\nu)\phi, \quad \forall \phi \in L^2(0,T;H_0^1(\Omega)).$$

From (3.1), (3.2) and Step 1 we can consider $R_{\omega,T} = \max\{h(s) : c_{\omega,T} \leq s \leq d\}$. Set $\varphi_\lambda(s) = se^{\lambda s^2}$ with $\lambda > \frac{R_{\omega,T}^2}{\alpha^2}$. We will also denote by $\varepsilon(\nu,n)$ any positive quantity such that

$$\lim_{\nu\to\infty} \limsup_{n\to+\infty} |\varepsilon(\nu,n)| = 0.$$

For $0 \leq \phi \in C_c^\infty(\Omega)$ we prove that

$$\int_0^T \langle (u_n)_t, \varphi_\lambda(u_n - u_\nu)\phi\rangle \geq \varepsilon(\nu,n). \tag{3.7}$$

Indeed, if we denote $\vartheta(s) = \displaystyle\int_0^s \varphi_\lambda(r)dr$ we obtain

$$\begin{aligned}
\int_0^T \langle (u_n)_t, \varphi_\lambda(u_n - u_\nu)\phi\rangle &= \int_0^T \langle (u_n - u_\nu)_t, \varphi_\lambda(u_n - u_\nu)\phi\rangle \\
&\quad + \int_0^T \langle (u_\nu)_t, \varphi_\lambda(u_n - u_\nu)\phi\rangle \\
&= \int_Q \frac{d}{dt}\vartheta(u_n - u_\nu)\phi + \int_0^T \langle (u_\nu)_t, \varphi_\lambda(u_n - u_\nu)\phi\rangle \qquad (3.8) \\
&= \int_\Omega \vartheta(u_n - u_\nu)(T)\phi - \int_\Omega \vartheta(u_n - u_\nu)(0)\phi + \int_0^T \langle (u_\nu)_t, \varphi_\lambda(u_n - u_\nu)\phi\rangle \\
&\geq \int_0^T \langle (u_\nu)_t, \varphi_\lambda(u_n - u_\nu)\phi\rangle\ .
\end{aligned}$$

On the other hand, we get

$$\begin{aligned}
\int_0^T \langle (u_\nu)_t, \varphi_\lambda(u_n - u_\nu)\phi\rangle &= \nu \int_Q (u - u_\nu)\varphi_\lambda(u_n - u_\nu)\phi \\
&= \nu \int_Q (u - u_\nu)\varphi_\lambda(u - u_\nu)\phi + \varepsilon(\nu,n)
\end{aligned}$$

since, for $n \to +\infty$, $\varphi_\lambda(u_n - u_\nu)$ converges to $\varphi_\lambda(u - u_\nu)$ $*$-weakly in $L^\infty(Q)$ and the other term is positive since the integrand function is positive. Therefore, we have

$$\int_0^T \langle (u_\nu)_t, \varphi_\lambda(u_n - u_\nu)\phi\rangle \geq \varepsilon(\nu,n),$$

and gathering together (3.8) with this inequality, we obtain (3.7).

Now, using (3.7) and $\varphi_\lambda(u_n-u_\nu)\phi$ as test function in (3.4) we obtain

$$\int_Q M(x,t,u_n)\nabla u_n\nabla(u_n-u_\nu)\varphi_\lambda'(u_n-u_\nu)\phi+\int_Q M(x,t,u_n)\nabla u_n\nabla\phi\,\varphi_\lambda(u_n-u_\nu)$$

$$+\int_Q g_n(x,t,u_n)\frac{|\nabla u_n|^2}{1+\frac{1}{n}|\nabla u_n|^2}\varphi_\lambda(u_n-u_\nu)\phi\le\int_Q f\,\varphi_\lambda(u_n-u_\nu)\phi-\varepsilon(\nu,n).$$

Moreover, choosing $\omega\subset\subset\Omega$ with $\operatorname{supp}\phi\subset\omega$ and since $u_n\to u$ weakly in $L^2(0,T;H_0^1(\Omega))$ and a.e. in Ω and for every $t\in(0,T)$, we deduce that $\varphi_\lambda(u_n-u_\nu)$ converges to $\varphi_\lambda(u-u_\nu)$ $*$-weakly in $L^\infty(Q)$, so that, by Egorov theorem

$$\int_Q f\,\varphi_\lambda(u_n-u_\nu)\phi-\int_Q M(x,t,u_n)\nabla u_n\cdot\nabla\phi\,\varphi_\lambda(u_n-u_\nu)\;=\varepsilon(\nu,n).$$

By the definition of $R_{\omega,T}$ we can state that

$$\int_Q g_n(x,t,u_n)\frac{|\nabla u_n|^2}{1+\frac{1}{n}|\nabla u_n|^2}\varphi_\lambda(u_n-u_\nu)\phi$$

$$\ge\int_{\omega\times(0,T)} g_n(x,t,u_n)\frac{|\nabla u_n|^2}{1+\frac{1}{n}|\nabla u_n|^2}\varphi_\lambda(u_n-u_\nu)\phi$$

$$\ge -R_{\omega,T}\int_Q|\nabla u_n|^2|\varphi_\lambda(u_n-u_\nu)|\phi\,.$$

Thus

$$\int_Q M(x,t,u_n)\nabla u_n\cdot\nabla(u_n-u_\nu)\varphi_\lambda'(u_n-u_\nu)\phi-R_{\omega,T}\int_Q|\nabla u_n|^2|\varphi_\lambda(u_n-u_\nu)|\phi$$

$$\le\varepsilon(\nu,n).$$

Adding

$$-\int_Q M(x,t,u_n)\nabla u\cdot\nabla(u_n-u_\nu)\varphi_\lambda'(u_n-u_\nu)\phi=\varepsilon(\nu,n)$$

on both sides of the previous inequality and since

$$\int_Q|\nabla u_n|^2|\varphi_\lambda(u_n-u_\nu)|\phi\;\le 2\int_Q|\nabla(u_n-u_\nu)|^2|\varphi_\lambda(u_n-u_\nu)|\phi$$

$$+2\int_Q|\nabla u|^2|\varphi_\lambda(u_n-u_\nu)|\phi=2\int_Q|\nabla(u_n-u_\nu)|^2|\varphi_\lambda(u_n-u_\nu)|\phi\;+\varepsilon(\nu,n),$$

we find, using also (1.2)

$$\int_Q|\nabla(u_n-u_\nu)|^2\Big[\alpha\varphi_\lambda'(u_n-u_\nu)-2R_{\omega,T}|\varphi_\lambda(u_n-u_\nu)|\Big]\phi\le\varepsilon(\nu,n).$$

Since $\lambda>\frac{R_{\omega,T}^2}{\alpha^2}$, it holds $\alpha\varphi_\lambda'(s)-2R_{\omega,T}|\varphi_\lambda(s)|\ge\dfrac{\alpha}{2}$ for every $s\in\mathbb{R}$ and we deduce (3.5).

Step 3. We proceed to show that the limit u of the approximated solutions u_n solves (1.4). We recall that u_n satisfies

$$\int_0^T\langle(u_n)_t,\varphi\rangle+\int_Q M(x,t,u_n)\nabla u_n\nabla\varphi+\int_Q g_n(x,u_n)\frac{|\nabla u_n|^2}{1+\frac{1}{n}|\nabla u_n|^2}\varphi=\int_Q f\varphi,$$

for any $\varphi \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(Q)$, $\varphi_t \in L^2(0,T;H^{-1}(\Omega))$, with $\varphi(T)=0$. The convergence of the two first terms is deduced as in the previous section. To finish the proof of Theorem 1.3 we only have to show that

$$\lim_{n\to+\infty} \int_Q g_n(x,t,u_n) \frac{|\nabla u_n|^2}{1+\frac{1}{n}|\nabla u_n|^2}\phi = \int_Q g(x,t,u)|\nabla u|^2\phi, \quad \forall \phi \in C_c^\infty(Q).$$

From Step 1, there exists $c_{\omega,T}>0$ such that $u_n(x) \geq c_{\omega,T} > 0$, a.e. $x \in \omega \equiv \operatorname{supp}\varphi$. Thus, we conclude that for some $c>0$ we have $|g_n(x,t,u_n(x))| \leq c$, a.e. $x\in\omega$ and for all $t \in (0,T)$. From (3.5) we deduce that there exists $\hbar_\omega \in L^2(\omega)$ such that $|\nabla u_n| \leq \hbar_\omega$ and ∇u_n converges to ∇u a.e. in ω and for all $t\in(0,T)$. Therefore,

$$|g_n(x,t,u_n(x))| \frac{|\nabla u_n(x)|^2}{1+\frac{1}{n}|\nabla u_n(x)|^2} \leq c\hbar^2_{\Omega_0}(x) \quad \text{a.e. } x\in\omega, \quad \forall t\in(0,T).$$

In addition, by the definition of g_n, for $n > 1/c_{\omega,T}$ we have $g_n(x,t,u_n(x)) = T_n\left(g(x,t,u_n(x))\right)$ and thus

$$g_n(x,t,u_n(x)) \frac{|\nabla u_n(x)|^2}{1+\frac{1}{n}|\nabla u_n(x)|^2} \longrightarrow g(x,t,u(x))|\nabla u(x)|^2 \quad \text{a.e. } x\in\omega, \quad \forall t\in(0,T).$$

By the Lebesgue dominated convergence theorem we deduce the desired limit.

For $s>0$ and $\varepsilon>0$, we take $T_{s+\varepsilon}(G_s(u_n))$ as test function in (3.4). Dropping positive terms, we deduce that

$$\int_Q g(x,t,u_n(x,t))|\nabla u_n(x,t)|^2 T_{s+\varepsilon}(G_s(u_n)) \leq \int_Q f + \int_Q u_0.$$

Let us observe that, by applying Fatou lemma, we deduce that

$$\int_Q g(x,t,u)|\nabla u|^2 \leq \int_Q f + \int_Q u_0.$$

Finally notice that, from the equation we deduce that $u_t \in L^2(0,T;H^{-1}(\Omega)) + L^1(Q)$, so that thank to a result of [29], we have $u\in C([0,T];L^1(\Omega))$, so that the initial datum is achieved, and Theorem 1.3 is proved. $\square$

Remark 3.1. Let us stress that condition (1.8) should be, in some sense, sharp to prove existence of a solution as suggested by the results in the stationary case (see [2]). In this paper the authors prove that, if $\gamma \geq 2$, then the elliptic boundary value problem associated to (1.1) do not admits in general solution. Specifically, in [2], the authors prove that, if $f\in L^q(\Omega)$, $q>\frac{N}{2}$, then finite energy solutions do not exist for the elliptic problem if either $\gamma>2$ or $\gamma=2$ with $\|f\|_{L^q(\Omega)} > C(\lambda_1)$. Where C is a positive constant depending on the first eigenvalue λ_1 of the laplacian. This strategy, due to a standard time-rescaling argument, is no longer available in the parabolic framework.

4. Comparison Principle and uniqueness result

From now on, we will focus our attention on quasilinear problems having natural growth in the gradient. We prove a comparison result which is new in the evolutive case by generalizing the elliptic argument of [4]. As a consequence we prove uniqueness for a fairly general class of problems and a stability result (see Section 5 below) as t goes to infinity.

We consider the problem

$$
\begin{cases} u_t - \alpha\Delta u + g(u)|\nabla u|^2 = f(x,t) & \text{in } Q, \\ u(x,0) = u_0(x) & \text{in } \Omega, \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{4.1}
$$

with $f \in L^1(Q)$ a nonnegative function, $u_0 \in L^1(\Omega)$, and g a Carathéodory function in I where $I =]0,b[$ (the value $b = +\infty$ is not excluded). We also assume that g is nonnegative and may be singular at the extremes of the interval.

Notice that, formally, from the above equation we expect the derivative of u to belong to the space $L^2(0,T;H^{-1}(\Omega)) + L^1(Q)$. To handle with this technicality we recall the following generalized integration by parts formula whose proof can be found in [17] (see also [13]).

Lemma 4.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous piecewise C^1 function such that $f(0) = 0$ and f' is zero away from a compact set of $\mathbb{R}$; let us denote $F(s) = \int_0^s f(r)dr$. If $u \in L^2(0,T,H_0^1(\Omega))$ is such that $u_t \in L^2(0,T;H^{-1}(\Omega)) + L^1(Q)$ and if $\psi \in C^\infty(\overline{Q})$, then we have*

$$
\begin{aligned} \int_0^T \langle u_t, f(u)\psi\rangle \; dt = &\int_\Omega F(u(T))\psi(T)\; dx \\ - \int_\Omega F(u(0))\psi(0)\; dx - &\int_Q \psi_t \; F(u)\; dxdt. \end{aligned} \tag{4.2}
$$

Observe that $u_t \in L^2(0,T;H^{-1}(\Omega)) + L^1(Q)$ implies that there exist $\eta_1 \in L^2(0,T;H^{-1}(\Omega))$ and $\eta_2 \in L^1(Q)$ such that $u_t = \eta_1 + \eta_2$. Obviously η_1 and η_2 are not uniquely determined but the integration by parts formula turn out to be independent on the representation of u_t once $\langle \cdot, \cdot \rangle$ indicate the duality between $H^{-1}(\Omega) + L^1(\Omega)$ and $H_0^1(\Omega) \cap L^\infty(\Omega)$.

Let us also recall that, because of a result in [29], a function $z \in L^2(0,T;H^1(\Omega))$ such that $z_t \in L^2(0,T;H^{-1}(\Omega)) + L^1(Q)$ turns out to belong to $C([0,T];L^1(\Omega))$, so all terms in (4.2) make sense.

Let us come back to our problem. We recall the definition of sub and supersolution.

Definition 4.2. We say that $z \in L^2(0,T;H^1(\Omega))$ such that $z_t \in L^2(0,T;H^{-1}(\Omega)) + L^1(Q)$ is a *subsolution* (respectively, *supersolution*) of problem (4.1) if $g(z)|\nabla z|^2 \in L^1(Q)$,

$$
g(z)|\nabla u|^2 \in L^1(Q), \quad f(\cdot, z) \in L^1(Q)
$$

and

$$
\int_0^T \langle z_t, w\rangle + \alpha \int\limits_Q \nabla z \cdot \nabla w + \int_Q g(z)|\nabla z|^2 w \overset{(\geq)}{\leq} \int\limits_Q f(x,t) w \, ,
$$

for every $w \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(Q)$, with $w \geq 0$.

A *solution* is a function which is both a subsolution and supersolution.

Now, we can prove the comparison principle. We assume the hypotheses

$$
t \mapsto e^{-\sigma(t)} \text{ is integrable in a right neighborhood of zero.} \tag{4.3}
$$

We fix a point $a \in I$, and define two auxiliary functions:

$$
\sigma(s) = \frac{1}{\alpha}\int_a^s g(r)dr, \quad s \in I,
$$

and

$$\varpi(s) = \int_0^s e^{-\sigma(r)}dr \quad s \in I.$$

Remark 4.3. Let us stress an important fact concerning of the model problem

$$\begin{cases} u_t - \alpha\Delta u + \dfrac{|\nabla u|^2}{u^\gamma} = f(x,t) & \text{in } Q, \\ u(x,0) = u_0(x) & \text{in } \Omega, \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{4.4}$$

Observe that if $\gamma \neq 1$ we have $e^{-\sigma(s)} = e^{-\frac{1}{\alpha}(s^{1-\gamma}-a^{1-\gamma})}$ while if $\gamma = 1$ is $e^{-\sigma(s)} = \left(\dfrac{a}{s}\right)^{\frac{1}{\alpha}}$. Therefore, the function is integrable on every interval containing 0 if and only if either $\alpha > 0$ and $0 < \gamma < 1$ or $\alpha > 1$ and $\gamma = 1$.

The following result may be proved in much the same way as Proposition 2.2 in [4].

Proposition 4.4. *Let u be a subsolution (respectively a supersolution) of* (4.1) *for which there exists $\delta \in]0,b[$ such that*

$$\begin{gathered} e^{-\sigma(T_\delta(u))}|\nabla T_\delta(u)| \in L^2(Q), \\ g(T_\delta(u))|\nabla T_\delta(u)|^2 e^{-\sigma(T_\delta(u))} \in L^1(Q), \end{gathered} \tag{4.5}$$

then the inequality

$$\int_0^T \langle u_t, e^{-\sigma(u)}w\rangle + \alpha\int_Q e^{-\sigma(u)}\nabla u\cdot\nabla w \overset{(\geq)}{\leq} \int_Q f(x,t)e^{-\sigma(u)}w \tag{4.6}$$

holds for every $w \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(Q)$, with $w \geq 0$.

Our comparison result is the following. For the sake of completeness we will state and prove it in a rather general case, namely for merely integrable data.

Theorem 4.5. *Let $f \in L^1(Q)$ be a nonnegative function and u, v be respectively a subsolution and a supersolution of problem* (4.1) *satisfying* (4.5) *and with initial data $u_0, v_0 \in L^1(\Omega)$ such that $u_0(x) \leq v_0(x)$ a.e. in Ω. Assuming that conditions* (4.3) *is fulfilled, if $u \leq v$ on $\partial\Omega \times (0,T)$ for all $t \in [0,T]$ (in the sense that $(u-v)^+ \in L^2(0,T;H_0^1(\Omega))$) then*

$$u(x,t) \leq v(x,t) \ \text{a.e in } \Omega, \quad \forall t \in [0,T].$$

Proof. Thanks to the assumption (4.5) we derive that $\varpi(u) \in L^2(0,T;H^1(\Omega))$ (analogously $\varpi(v) \in L^2(0,T;H^1(\Omega))$). So that, $[\varpi(u)-\varpi(v)]^+ \in L^2(0,T;H_0^1(\Omega))$ since $u \leq v$ on $\partial\Omega$ and ϖ is increasing. Therefore $T_k[\varpi(u)-\varpi(v)]^+ \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(Q)$. Applying Proposition 4.4, with $w = T_k[\varpi(u)-\varpi(v)]^+$, we obtain

$$\begin{aligned} &\int_0^T \langle \varpi(u)_t, T_k[\varpi(u)-\varpi(v)]^+\rangle + \alpha\int_Q e^{-\sigma(u)}\nabla u\nabla T_k[\varpi(u)-\varpi(v)]^+ \\ &\leq \int_Q f(x,t)e^{-\sigma(u)}T_k[\varpi(u)-\varpi(v)]^+. \end{aligned} \tag{4.7}$$

Similarly,

$$\begin{aligned}&\int_0^T \langle \varpi(v)_t, T_k[\varpi(u)-\varpi(v)]^+\rangle + \alpha\int_Q e^{-\sigma(v)}\nabla v \nabla T_k[\varpi(u)-\varpi(v)]^+\\ &\geq \int_Q f(x,t)e^{-\sigma(v)}T_k[\varpi(u)-\varpi(v)]^+.\end{aligned}\tag{4.8}$$

Using that $u_0 \leq v_0$, and Lemma 4.1 we get

$$\begin{aligned}&\int_0^T \langle \varpi(u)_t, T_k[\varpi(u)-\varpi(v)]^+\rangle - \int_0^T \langle \varpi(v)_t, T_k[\varpi(u)-\varpi(v)]^+\rangle\\ &= \int_0^T \langle (\varpi(u)-\varpi(v))_t, T_k[\varpi(u)-\varpi(v)]^+\rangle = \int_\Omega \Theta_k(\varpi(u)-\varpi(v))^+(T)\,,\end{aligned}$$

where $\Theta_k(s) = \displaystyle\int_0^s T_k(r)dr$.

Hence, if we substract (4.8) from (4.7), and use the above equality we derive

$$\begin{aligned}&\int_\Omega \Theta_k(\varpi(u)-\varpi(v))^+(T) + \alpha\int_Q \left(e^{-\sigma(u)}\nabla u - e^{-\sigma(v)}\nabla v\right)\nabla T_k[\varpi(u)-\varpi(v)]^+\\ &\leq \int_Q f(x,t)(e^{-\sigma(u)}-e^{-\sigma(v)})T_k[\varpi(u)-\varpi(v)]^+.\end{aligned}$$

Observing that the function ϖ is increasing ($\varpi'(s) = e^{-\sigma(s)} > 0$) and $f \geq 0$, it follows that

$$f(x,t)(e^{-\sigma(u)}-e^{-\sigma(v)})T_k[\varpi(u)-\varpi(v)]^+ \leq 0$$

a.e. $x \in \Omega$. Consequently, we have

$$\begin{aligned}&\int_\Omega \Theta_k(\varpi(u)-\varpi(v))^+(T)\\ &+ \alpha\int_Q \left(e^{-\sigma(u)}\nabla u - e^{-\sigma(v)}\nabla v\right)\nabla T_k[\varpi(u)-\varpi(v)]^+ \leq 0,\end{aligned}$$

i.e.

$$\int_\Omega \Theta_k(\varpi(u)-\varpi(v))^+(T) + \alpha\int_Q \left|\nabla T_k[\varpi(u)-\varpi(v)]^+\right|^2 \leq 0.$$

Therefore, because of the definition of $\Theta_k(s)$ we get

$$[\varpi(u)-\varpi(v)]^+(T) = 0, \quad \forall k > 0.$$

Since T is arbitrary, we conclude that

$$u(x,t) \leq v(x,t) \text{ a.e. in } \Omega, \quad \forall t \in [0,T].$$

□

Summarizing, we have the desired uniqueness result.

Theorem 4.6. *Problem* (4.1) *has at most a solution satisfying* (4.5) .

Recalling Remark 4.3, in the model case we get the following.

Corollary 4.7. *Problem* (4.4) *has at most one solution satisfying* (4.5) *provided that either* $\gamma < 1$ *and* $\alpha > 0$ *or that* $\gamma = 1$ *and* $\alpha > 1$.

5. Asymptotic behavior

In this section, as an application of the results of the previous sections, we prove the asymptotic behavior as t tends to $+\infty$ of solutions u of the problem (4.1) with $f(x,t) = f(x)$ satisfying (1.5), $u_0 \in L^\infty(\Omega)$ is a nonnegative function and $g(s)$ satisfies (1.7), (4.3), (4.5) and there exists $r_0 >> 1$ such that

$$g(rs) \geq \frac{g(s)}{r}, \tag{5.1}$$

for any $s \geq 0$ and $r > r_0$. For the sake of simplicity, we will denote by Q_∞ the infinite cylinder $\Omega \times (0, \infty)$.

Our result generalizes the one in [23] where the authors consider a bounded nonlinearity $g(s)$. The techniques we use are a readaptation of the ones introduced in [28].

A solution for problem (4.1) exists and is unique as proved, respectively, in Theorem 1.3 and Theorem 4.6.

Observe that, if $f \in L^q(\Omega)$, with $q > \frac{N}{2}$, then, standard regularity results (see [6]) implies that the solutions of problem (4.1) are bounded in Q.

We will prove that, as t tends to $+\infty$, $u(x,t)$ converges to $v(x)$ which is the unique solution of the elliptic problem (see ([1],[2],[3] for the existence result and [4] for the uniqueness result)

$$\begin{cases} -\Delta v + g(v)|\nabla v|^2 = f & \text{in } \Omega\,, \\ v(x) = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{5.2}$$

such that $g(v)|\nabla v|^2 \in L^1(\Omega)$ and $v > 0$ a.e. on Ω.

We can now formulate our result.

Theorem 5.1. *Assume that $f \in L^q(\Omega)$ with $q > \frac{N}{2}$ is a strictly positive function on any compact set contained in Ω and $u_0 \in L^\infty(\Omega)$ is a nonnegative function. If g satisfies (4.5), then $u(x,t)$, the weak solution of problem (4.1), satisfies*

$$\lim_{t\to+\infty} u(x,t) = v(x) \qquad \textit{a.e. in } \Omega \textit{ and } *\textit{-weakly } L^\infty(\Omega).$$

Remark 5.2. Notice that the result of Theorem 5.1 implies that $u(x,t)$ converges to the stationary solution in $L^q(\Omega)$, for any $q \geq 1$. As we also noticed, assumption (4.3) implies that the result holds true in the model case $g(s) = \frac{1}{s^\gamma}$ only if $\gamma < 1$. However, the same proof can be readapted for the model problem

$$\begin{cases} u_t - \alpha\Delta u + \dfrac{|\nabla u|^2}{u} = f(x,t) & \text{in } Q_\infty, \\ u(x,0) = u_0(x) & \text{in } \Omega, \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,\infty). \end{cases}$$

if $\alpha > 1$ (see also Remark 4.3).

Proof of Theorem 5.1. We will split the proof into a few steps. Namely:

- Step 1. We prove the result by exploiting the monotonicity character of the solution with respect to t with particular initial data. Concretely:
  - Substep 1.1. The solution is nondecreasing if $u_0 = 0$.
  - Substep 1.2. The solution is nonincreasing if $u_0 = rv$, for any $r > 1$.
  - Substep 1.3. Passage to the limit and proof of the result.

- Step 2. By a comparison argument, we prove the result for initial data u_0 such that $0 \le u_0 \le rv$ for any $r > 1$.
- Step 3. We perform a truncation argument to achieved the result in the general case of $u \in L^\infty(\Omega)$.

Step 1. Monotonicity of the Solution. Proof completed for particular initial data.

Substep 1.1. $u_0 = 0$. Let w be the solution of (4.1) with $u(x, 0) = 0$.
We will prove that $w(x,t)$ is nondecreasing with respect to t. Fix $T > 0$ and let us introduce for any $\eta > 1$ the sequence of functions in Q defined by

$$w^\eta(x,t) = w(x, \eta + t)\,.$$

Recalling that f does not depend on time, we deduce that $w^\eta(x,t)$ solves

$$\begin{cases} w^\eta_t - \Delta w^\eta + g(w^\eta)|\nabla w^\eta|^2 = f & \text{in } Q, \\ w^\eta(x,0) = w(x,\eta) & \text{in } \Omega, \\ w^\eta(x,t) = 0 & \text{on } \partial\Omega \times (0,T)\,. \end{cases}$$

Since $w(x,\eta) \ge 0$, we observe that w^η is a supersolution of the problem

$$\begin{cases} w_t - \Delta w + g(w)|\nabla w|^2 = f(x) & \text{in } Q, \\ w(x,0) = 0 & \text{in } \Omega, \\ w(x,t) = 0 & \text{on } \partial\Omega \times (0,T)\,. \end{cases}$$

Applying Theorem 4.5 we get $w(x,t) \le w^\eta(x,t)$. This implies that $w^\eta(x,t)$ is increasing in η.

Moreover, $v(x)$ is a supersolution of the above problem, then applying again Theorem 4.5 we deduce

$$w(x,t) \le v(x), \quad \text{a.e. in } \Omega \qquad \forall t \in (0,T). \tag{5.3}$$

Therefore there exists a function $\widetilde{w}(x)$ such that $w^\eta(x,t)$ converges a.e. in Q to $\widetilde{w}(x)$ as η tends to $+\infty$. The fact that $\tilde{w}$ does not depend on time follows by the inequality

$$w^\eta(x,0) = w(x,\eta) \le w(x,\eta+t) = w^\eta(x,t) \le w(x,T)\,.$$

Moreover, thanks to (5.3), since v is bounded, we have that

$$w^\eta(x,t) \longrightarrow \widetilde{w}(x) \qquad \text{a.e. and } *\text{-weakly in } L^\infty(Q)\,.$$

Substep 1.2. $u_0 = rv$, $r > 1$. Nonincreasing solutions.
Suppose that z is the solution of the following problem:

$$\begin{cases} z_t - \Delta z + g(z)|\nabla z|^2 = f & \text{in } Q \\ z(x,0) = rv(x) & \text{in } \Omega \\ z(x,t) = 0 & \text{on } \partial\Omega \times (0,T)\,, \end{cases}$$

where $r > 1$ is a fixed number.
Observe that, for $r > r_0$, $rv(x)$ is a supersolution of the above problem. Indeed, assumption (5.1) implies

$$\frac{d}{dt}(rv(x)) - \Delta(rv) + g(rv)|\nabla(rv(x))|^2 \ge r(-\Delta v + g(v)|\nabla v|^2) \ge f.$$

Now, since

$$rv(x) \geq 0 \text{ on } \Omega \times (0,\infty)\,,$$

using again Theorem 4.5, we deduce

$$v(x) \leq z(x,t) \leq rv(x) \quad \text{ a.e. in } \quad Q\,. \tag{5.4}$$

Moreover, we can compare $z(x,t)$ with $z(x,t+s)$ for $s \in \mathbb{R}^+$. In fact, we observe that $z(x,t)$ is the solution of the problem (4.1) which for $t = 0$ achieves the value $rv(x)$, while $z(x,t+s)|_{t=0} = z(x,s)$. Since $rv(x)$ is a supersolution and $rv(x) \geq z(x,s)$, then by the Theorem 4.5 we deduce that $z(x,t) \geq z(x,t+s)$ a.e. on Q. Let us consider now the sequence of problems

$$\begin{cases} z^\tau_t - \Delta z^\tau + g(z^\tau)|\nabla z^\tau|^2 = f & \text{in } Q \\ z^\tau(x,0) = z(x,\tau) & \text{in } \Omega \\ z^\tau(x,t) = 0 & \text{on } \partial\Omega \times (0,T)\,, \end{cases} \tag{5.5}$$

$\forall \tau \in \mathbb{N}$. As before, the monotonicity in t of $z(x,t)$ implies the monotonicity of $z^\tau(x,t)$ with respect to τ. Hence, $z^\tau(x,t)$ converges a.e. to a function $\overline{z}(x,t)$ and arguing as in the previous step we can deduce that $\overline{z}$ does not depend on t.

Now, it is easy to readapt the proof of Theorem 1.3 and to prove that w^η (resp. z^τ) converges to $\overline{w}(x)$ (resp. $\overline{z}(x)$) strongly in $L^2(0,T;H^1_{loc}(\Omega))$, and $g(w^\eta)|\nabla w^\eta|^2$ (resp. $g(z^\tau)|\nabla z^\tau|^2$) converges to $g(\overline{w})|\nabla\overline{w}|^2$ (resp. $g(\overline{z})|\nabla\overline{z}|^2$) strongly in $L^1_{loc}(Q)$.

So we can pass to the limit in the weak formulation of both problem, which verifies w^η and z^τ to conclude that both $\tilde{w}(x)$ and $\overline{z}(x)$ are stationary solution of (4.1), and applying the uniqueness result for the elliptic problem (5.2) in [4] we deduce that $\tilde{w}(x) \equiv \overline{z}(x) \equiv v(x)$.

To conclude, we will perform the passage to the limit for w^η being the one for z^τ identical.

Substep 1.3. Passage to the limit and proof of the result.

Consider the weak formulation of (5) and choose $\Psi(x) \in C^\infty_0(\Omega)$ as test function to obtain

$$\int_0^T \langle w^\eta_t, \Psi(x)\rangle + \int_Q \nabla w^\eta \nabla\Psi(x) + \int_Q g(w^\eta)|\nabla w^\eta|^2\Psi(x) = \int_Q f\Psi(x)\,.$$

Now, since Ψ does not depend on t, and both $w^\eta(x,T)$ and $w^\eta(x,0)$ admit the same limit, we have

$$\begin{aligned} &\lim_{\eta\to\infty}\int_0^T \langle w^\eta_t, \Psi(x)\rangle \\ &= \lim_{\eta\to\infty}\left(-\int_0^T \langle w^\eta, \Psi(x)_t\rangle + \int_\Omega w^\eta(x,T)\Psi(x) - \int_\Omega w^\eta(x,0)\Psi(x)\right) = 0. \end{aligned}$$

So that, thanks to the strong compactness of w^η in $L^2(0,T;H^1_{loc}(\Omega))$ and of $g(w^\eta)|\nabla w^\eta(x)|^2$ in $L^1(Q)$, we then obtain

$$\int_\Omega \nabla\overline{w}(x)\cdot\nabla\Psi(x) + \int_\Omega g(\overline{w})|\nabla\overline{w}(x)|^2\Psi(x) = \int_\Omega f\Psi(x)$$

and then, since by an easy density argument we can take $\Psi(x) \in H^1_0(\Omega) \cap L^\infty(\Omega)$,

$$\overline{w}(x) \equiv v(x)\,,$$

where $v(x)$ is the unique solution of problem (5.2).

Observe that, (5.3) and (5.4), the a.e. convergence of both $w(x,t)$ and $z(x,t)$ to $v(x)$ and the boundedness of $v(x)$ actually yields that the convergences to the stationary solution are in $*$-weakly in $L^\infty(\Omega)$.

Step 2. Proof completed if $0 \le u_0 \le rv$.

Now let us consider an initial datum $u_0(x)$ such that $0 \le u_0(x) \le rv(x)$ for some $r \ge 1$, and let $u(x,t)$ be the solution of problem (4.1).

Thanks to Theorem 4.5 we have that

$$w(x,t) \le u(x,t) \le z(x,t) \quad \text{a.e. in } Q\,.$$

So, passing to the limit with respect to the t variable, we have that

$$v(x) = \lim_{t\to+\infty} w(x,t) \le \liminf_{t\to\infty} u(x,t)$$

$$\le \limsup_{t\to\infty} u(x,t) \le \lim_{t\to+\infty} z(x,t) = v(x).$$

Then the limit with respect to the t variable of $u(x,t)$ exists, coincides with $v(x)$ and the convergence is $*$-weak $L^\infty(\Omega)$.

Step 3. $u_0 \in L^\infty(\Omega)$.

Let us define the monotone nondecreasing (in τ) family of functions

$$u_{0,\tau} = \min(u_0, \tau v).$$

For every fixed $\tau > 1$, we consider the solution $u_\tau(x,t)$ of problem (4.1) with $u_{0,\tau}$ as initial datum. As we have shown in the previous step $u_\tau(x,t)$ converges to v a.e. in Ω, as t tends to infinity. Thanks to *Lebesgue dominated convergence theorem*, and to the fact that $v > 0$ a.e. on Ω, we can easily check that $u_{0,\tau}$ converges to u_0 in $L^2(\Omega)$ as τ tends to infinity.

We can easily prove, for any fixed $\tau > 1$, the following estimate (see the proof of Theorem 4.5 applied to u and u_τ)

$$\int_\Omega |u - u_\tau|^2(t) \ \le \int_\Omega |u_0 - u_{0,\tau}|^2,$$

for every $t > 0$. Therefore, we have

$$\|u(x,t) - v(x)\|_{L^2(\Omega)} \le \|u(x,t) - u_\tau(x,t)\|_{L^2(\Omega)} + \|u_\tau(x,t) - v(x)\|_{L^2(\Omega)}.$$

Since the previous estimate in is uniform in t, for every fixed ε, we can choose $\bar\tau$ large enough such that

$$\|u(x,t) - u_{\bar\tau}(x,t)\|_{L^2(\Omega)} \le \frac{\varepsilon}{2}, \qquad \text{a.e. in } \Omega$$

for every $t > 0$. On the other hand, since the result is true for initial datum between 0 and $\overline{\tau} v$, then there exists $\bar t$ such that

$$\|u_{\bar\tau}(x,t) - v(x)\|_{L^2(\Omega)} \le \frac{\varepsilon}{2}, \text{ a.e. in } \Omega$$

for every $t > \bar t$, and this proves our result because of the boundedness of $u(x,t)$. $\square$

Now, we give the idea of the proof of the result described in Section 2 for the asymptotic behavior of the solutions of (1.1).

Sketch of the proof of Theorem 2.2. We just give an outline of the proof since the idea is mainly the same as the one in the proof of Theorem 5.1. We divide the proof into three steps.

- Step 1. We prove the result with particular initial data and $f \in L^\infty(\Omega)$. Concretely:
  - Substep 1.1. The solution is nondecreasing if $u_0 = 0$.
  - Substep 1.2. The solution is nonincreasing if $u_0 = k < \kappa$, with $k > h^{-1}(\|f\|_{L^\infty(\Omega)})$.
  - Substep 1.3. Passage to the limit and proof of the result if $0 \le u_0 \le k < \kappa$.
- Step 2. For initial data $u_0 < \kappa$ with $f \in L^\infty(\Omega)$.
- Step 3. We follow a truncation argument to prove the result for $f \in L^1(\Omega)$.

Substep 1.1. It works exactly as in the proof of Theorem 5.1.

Substep 1.2. We consider the same argument as in Theorem 5.1 by defining the related problem solved by $z(x,t)$ with initial datum $u_0 = k$. Thanks to the ellipticity and the fact that h in nondecreasing, we check that k is a supersolution for the parabolic problem and so $0 \le z(x,t) \le k$. By comparison, we can prove the result for every $0 \le u_0 \le k$.

Substep 1.3. As in the proof of Theorem 5.1, we pass to the limit as t goes to $+\infty$ and we obtain the desired result.

Step 2. We take $k < \kappa$, $u_{0,k} = T_k(u_0)$ and $u_k(x,t)$ the solution with initial data $u_{0,k}$. The result is true for u_k by the previous step, and, reasoning as in the proof of Theorem 5.1, we get a stability principle in $L^2(\Omega)$, that is, there exists $C > 0$ such that

$$\|u(x,t) - u_k(x,t)\|_{L^2(\Omega)} \le C\|u_0 - u_{0,k}\|_{L^2(\Omega)} \quad \forall t \in (0,T).$$

So that, we can fix $\overline{k}$ such that

$$\|u(x,t) - u_{\overline{k}}(x,t)\|_{L^2(\Omega)} \le \frac{\varepsilon}{2},$$

and we take t large enough such that

$$\|u_{\overline{k}}(x,t) - v(x)\|_{L^2(\Omega)} \le \frac{\varepsilon}{2}.$$

We conclude by using the triangular inequality

$$\|u(x,t) - v(x)\|_{L^2(\Omega)} \le \|u(x,t) - u_k(x,t)\|_{L^2(\Omega)} + \|u_k(x,t) - v(x)\|_{L^2(\Omega)}.$$

Step 3. We truncate f and we take the solutions for u_k and for v_k. We choose k_0 such that

$$\|u(x,t) - u_{k_0}(x,t)\|_{L^2(\Omega)} \le \frac{\varepsilon}{3}, \qquad \|v_{k_0}(x) - v(x)\|_{L^2(\Omega)} \le \frac{\varepsilon}{3},$$

and t such that

$$\|u_{k_0}(x,t) - v_{k_0}(x)\|_{L^2(\Omega)} \le \frac{\varepsilon}{3}.$$

This finishes the sketch of the proof. □

Remark 5.3. Let us just notice that, a larger class of singular problems are in order to be considered. Focussing on the lower order term, we suggest, as an open problem, the study of the class of problems whose model is

$$
\begin{cases} u_t - \Delta u + \dfrac{u|\nabla u|^q}{|\kappa - u|^\gamma(\kappa - u)} = f(x,t) & \text{in } Q, \\ u(x,0) = u_0(x) & \text{in } \Omega, \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T)\,, \end{cases} \tag{5.6}
$$

where q, κ, γ are nonnegative parameters. As we mentioned above in this paper we started in a twofold direction by considering, in some sense, two extreme cases of problem (5.6), namely the case $q = 0, \kappa > 0, \gamma > 1$ (semilinear case) and $\kappa = 0, q = 2, \gamma < 2$ (quasilinear problems with *natural growth*). Most of the remaining problems are far to be well understood even in the stationary framework.

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Pedro J. Martínez-Aparicio, Departamento de Análisis Matemático, Campus Fuentenueva S/N, Universidad de Granada 18071 -Granada, Spain
E-mail address: pedrojma@ugr.es

Francesco Petitta, CMA, University of Oslo, P.O. Box 1053 Blindern, NO-0316 Oslo, Norway
E-mail address: francesco.petitta@cma.uio.no